

To what extent can a solar transit's light
curve be modeled using only the star's
occult area?

Math AA HL IA

September 2024 - January 2025

Introduction	1
Problem formulation	2
Mathematical development	3
Interpretation of the results	15
Limitations	17
Conclusion	18
References	19
Appendix	22
Appendix 1	22
Appendix 2	22
Appendix 3	24

Introduction

Eclipses have fascinated humanity for centuries, inspiring cultural interpretations and scientific exploration. Ancient civilizations meticulously recorded these celestial events, often imbuing them with religious significance. Some ancient observations in China have even contributed to modern studies on Earth's rotational changes (*History of Eclipses - NASA Science*, n.d.). Today, astronomers use eclipses to study phenomena like the Sun's corona and Earth's atmosphere, highlighting their enduring scientific importance. As science advanced and humans looked at the Sun with better equipment, it has been found that the planets Mercury and Venus also engage in eclipses. These phenomena are usually referred to as transits but essentially describe similar events for celestial bodies. Motivated by a love for astronomy and the recent purchase of a sun filter for the school's small telescope, this essay seeks to mathematically model the visible area of a star during transit and find a relationship between it and the light received on Earth. Fostering a better understanding of this phenomenon can bring insight into practical applications such as optimizing solar energy production, understanding wildlife behavior, and contextualizing the cultural significance of eclipses while also developing mathematical skills. (Adkins, 2024)

Problem formulation

Before diving into mathematics, we must understand the nature of the issue for which we seek a mathematical solution. Transits are an astronomical phenomenon that occurs, from a viewer's perspective, when one astronomical object "passes by" in front of another. Eclipses are a typical example of a transit.

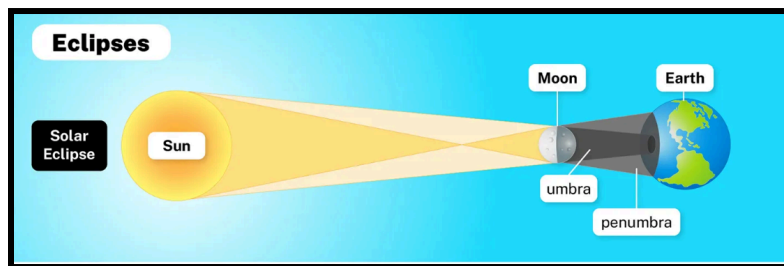


Diagram 1 - Sourced from (The Editors of Encyclopaedia Britannica, n.d.).

A brightness-over-time graph can describe this phenomenon from the observer's perspective. In exoplanet studies, this is often called a "light curve," which is used to detect them (Caltech, n.d.). The light curve depends on many different properties of the star and the planet. This essay aims to mathematically model the sun's visible or not visible area during a Venus transit to compare it to a light curve to see the similarity between quantities and better understand the properties that influence the shape of the light curve.

Mathematical development

Firstly, the investigation of the area can be divided into different parts. For the first mathematical model, the essay will consider perfectly spherical objects transiting before a perfectly spherical sun. From this, the light curve can be separated into three main inflection points, labeled 1, 2, 3, and 4 in diagram 2, which shows a transit from the viewer's perspective.

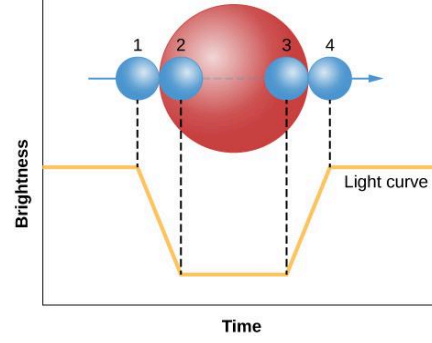


Diagram 2. Sourced from (Libretexts, 2022)

The parts before 1 and after 4 of the light curve are the longest and simplest. It represents the star's brightness when nothing is transiting it.

Part 1 to 2 of the light curve represents the point in which only a portion of the object's area is occulting the light source, at this moment occulting more of the sun as it moves.

Part 2 to 3 represents the planet's transit through the star once all of its body is casting a shadow on the third observer.

Part 3 to 4 is much like 1 to 2, but instead of decreasing, the light increases, representing the shadow leaving the star.

To model this, two circles will be defined to represent the star and the planet. These have the following equations:

$$\text{Star: } r_{star}^2 = x^2 + y^2, \text{ for } \{-r_{star} < x < r_{star}\}$$

$$\text{Planet: } r_{planet}^2 = (x + p)^2 + y^2, \text{ for } \{p - r_{planet} < x < p + r_{planet}\}$$

Where r_{star} is the angular radius of the star and r_{planet} is the angular radius of the transiting object. This one can be a planet or not, but it is named after one due to the origin of light curves being in the field of exoplanet study. Meanwhile p is the object's position, and for now, we assume that the transiting object's path follows a straight line through the star's center.

Following this logic, the two flat line states are the easiest to understand regarding the function of the area and its domain.

$$A_{max} = \pi r_{star}^2 \text{ for } \{p \leq -(t + r), p \geq t + r\}$$

$$A_{min} = \pi r_{star}^2 - \pi r_{planet}^2 \text{ for } \{t - r \leq p \leq r - t\}$$

Where A_{max} is the maximum area visible in the system, meaning the part previously labeled as part 4 to 1, and A_{min} is the minimum area visible in the system in part 2 to 3. We also have to add the cases in which the occulting object is bigger than the star, for which we will get 0 area visible.

$$A_{occultation} = 0 \text{ for } \{r_{star} - r_{planet} < p < r_{planet} - r_{star}\}$$

In the points where the object partially transits the star, not all of its area blocks the star's light. In this case, integration can be used to find the areas with blocked light. To do this, it is necessary to consider the two solutions of the circles' equations, which will share the same x-value.

$$\pm \sqrt{r_{star}^2 - x^2} = \pm \sqrt{r_{planet}^2 - (x + p)^2}$$

$$\text{Solving for } x = \frac{(-r_{star}^2 - p^2 + r_{planet}^2)}{2p}, \text{ from now onwards, referred to as } k = \frac{(-r_{star}^2 - p^2 + r_{planet}^2)}{2p}.$$

This gives the x-position of the vertical line in Diagram 3. Using this line, we can separate four areas that will be integrated, as shown in Diagram 3.

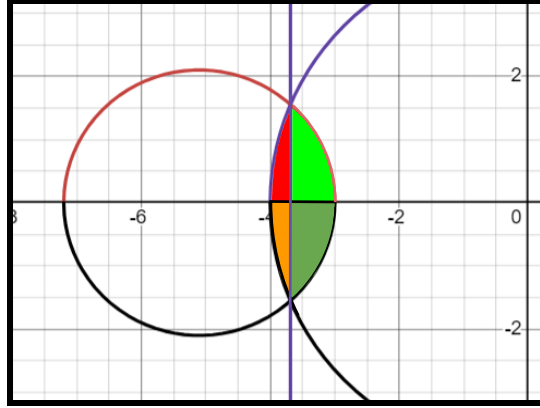


Diagram 3. Desmos, own source.

$$\begin{aligned} \text{Red: } \int_{-r_{star}}^k (+\sqrt{r_{star}^2 - x^2})dx & \quad \text{Bright green: } \int_k^{r_{planet}+p} (+\sqrt{r_{planet}^2 - (x-p)^2})dx \\ \text{Orange: } \int_{-r_{star}}^k (-\sqrt{r_{star}^2 - x^2})dx & \quad \text{Dark green: } \int_k^{r_{planet}+p} (-\sqrt{r_{planet}^2 - (x-p)^2})dx \end{aligned}$$

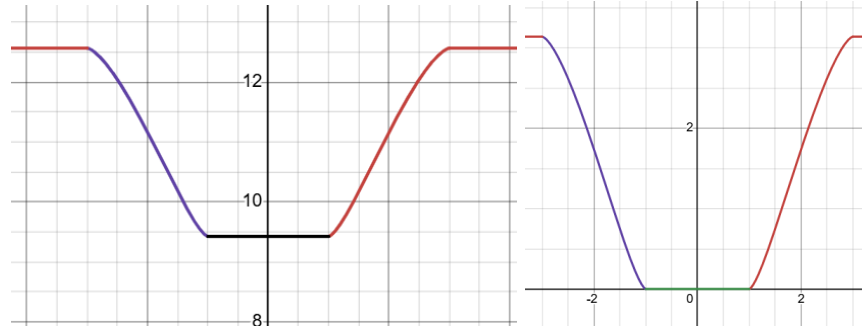
This gives the areas for the parts of the star that get hidden because of the planet transiting while it is coming from the left. If everything is put together and simplified, it looks like this:

$$\begin{aligned} A = \pi r^2 - 2 * \int_{-r_{star}}^k (+\sqrt{r_{star}^2 - x^2})dx - 2 * \int_k^{t+p} (+\sqrt{r_{planet}^2 - (x-p)^2})dx \\ \text{for } \{- (r_{star} + r_{planet}) < p < r_{planet} - r_{star}\} \end{aligned}$$

The non-simplified step is available in Appendix 1. The same logic can be used to reach the integration for the area while the planet is leaving through the right, using different limits for the definite integrals.

$$\begin{aligned} A = \pi r^2 - 2 * \int_k^{r_{star}} (+\sqrt{r_{star}^2 - x^2})dx - 2 * \int_{p-r_{planet}}^k (+\sqrt{r_{planet}^2 - (x-p)^2})dx \\ \text{for } \{r_{star} + r_{planet} < p < r_{star} - r_{planet}\} \end{aligned}$$

If the area visible is now plotted against the change in position of the planet, meaning $f(p) = A$, some values are added for the radii of both the planet and the star, the area/position curve shows up.



Left: Diagram 4. The area/position graph when $r_{planet} = 1$ and $r_{star} = 2$.

Right: Diagram 5. The area/position graph when $r_{planet} = 2$ and $r_{star} = 1$.

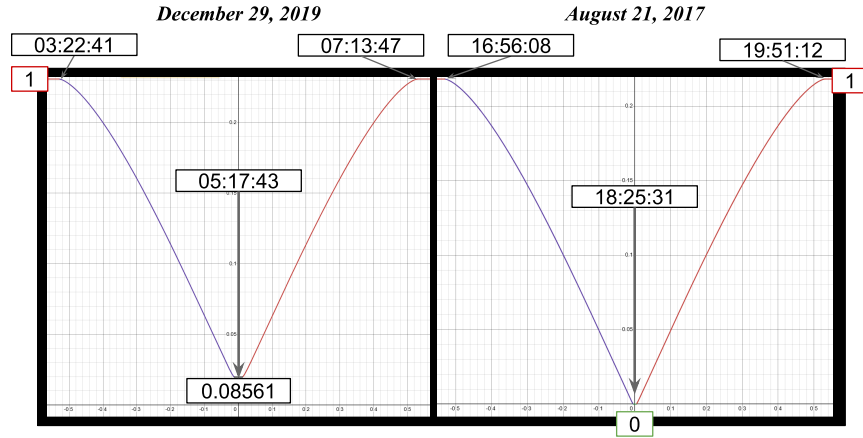
One of the first things that stands out is the symmetry of transits and eclipses. They are symmetrical over the y-axis, which will be helpful in the study.

The annular eclipse of December 26, 2019, and the total eclipse of August 21, 2017, will be modeled. Both eclipses will be modeled after how they were seen from the point of greatest eclipse on Earth. The star's movement is described as linear using a small-angle approximation.

Looking at (Espenak, n.d.) and (Espenak, n.d.-b), the following data and the point of greatest eclipse can be obtained:

For the annular eclipse on <i>December 26, 2019:</i>	And for the total eclipse on <i>August 21, 2017:</i>
$r_{sun} = 16'15.7'' = 0.271028^\circ$	$r_{sun} = 15'48.7'' = 0.263528^\circ$
$r_{moon} = 15'33.0'' = 0.259167^\circ$	$r_{moon} = 16'03.4'' = 0.267611^\circ$

The following area-position curves appear when these angles are introduced to the model. The labels mark the UTC times the eclipse starts, ends, and reaches a maximum (Stellarium Contributors) and the relative variation in the area visible.



Diagrams 6 and 7. Area-position graph of the annular eclipse, Area-position graph of the total eclipse.

However, many transits and eclipses do not follow a straight line through the star's center. Because of this, the model can be expanded to fit those other cases. In this case, the star's center will be retaken as the origin of coordinates, while the other circle will gain a new variable for the angular distance from the x-axis at which it is transiting through the star h .

$$\text{Hence, the star will remain represented by: } y = \pm \sqrt{r_{star}^2 - x^2}$$

$$\text{Nevertheless, the planet is now represented by: } y = \pm \sqrt{r_{planet}^2 - (x - p^2) + h}$$

Allowing for the modeling of positions such as this one:

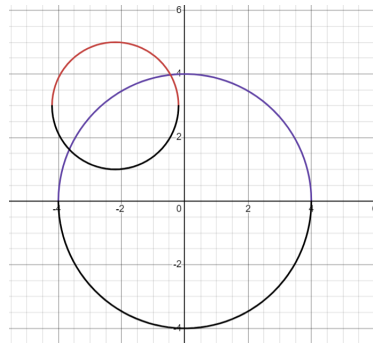


Diagram 8.

The integration and limits previously found will no longer work. The solutions between the two circles will be important for setting the definite integrals.

$$\pm \sqrt{r_{planet}^2 - (x - p)^2} + h = \pm \sqrt{r_{star}^2 - x^2}$$

Which will have two solutions x_1 x_2 . By simplifying the equations and using the quadratic formula, one can find the solutions to where the two circles intersect. This process is shown in Appendix 2. The two solutions are:

$$x_1 = \frac{-p(r_{planet}^2 - p^2 - r_{star}^2 - h^2) - \sqrt{(p(r_{planet}^2 - p^2 - r_{star}^2 - h^2))^2 - (p^2 + h^2)((r_{planet}^2 - p^2 - r_{star}^2 - h^2)^2 - 4h^2 r_{star}^2)}}{2(p^2 + h^2)}$$

$$x_2 = \frac{-p(r_{planet}^2 - p^2 - r_{star}^2 - h^2) + \sqrt{(p(r_{planet}^2 - p^2 - r_{star}^2 - h^2))^2 - (p^2 + h^2)((r_{planet}^2 - p^2 - r_{star}^2 - h^2)^2 - 4h^2 r_{star}^2)}}{2(p^2 + h^2)}$$

As the planet moves through the star, the integration rules will change in many separate phases and domains. The June 2012 Venus transit will be the studied scenario. Its path was recorded as not going through the center.

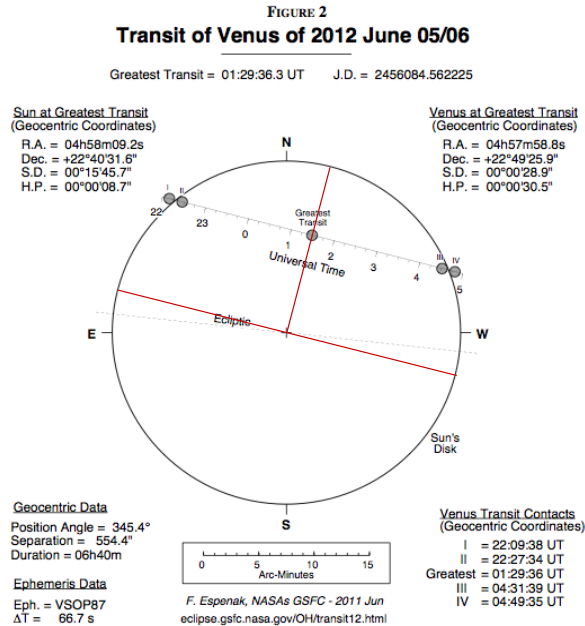


Diagram 9. Source: (NASA - 2012 Transit of Venus, n.d.)

According to Diagram 9, the angular radius of Venus that day was 0.008 degrees, and the sun's was 0.2622 degrees. The value h is less straightforward, but the data gives the coordinates of the sun's and Venus's centers at the time of greatest transit, whose difference is equal to the value h described.

$$\text{Sun RA: } 04h58m09.2s = 74.538^\circ \mid \text{Sun Dec: } + 22^\circ40'31.6'' = 22.675^\circ$$

$$\text{Venus RA: } 04h 57m 58.8s = 74.495^\circ \mid \text{Venus Dec: } + 22^\circ49'25.9'' = 22.824^\circ$$

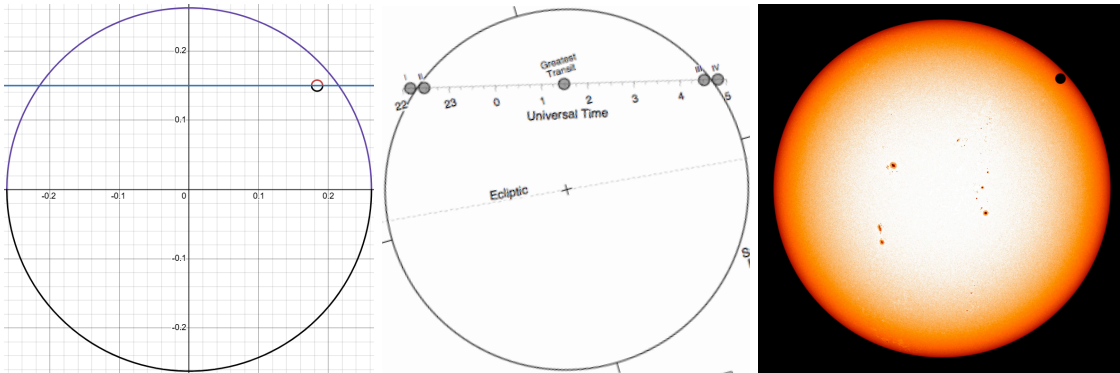
Meaning that the vector of their angular distance is

$$\rightarrow (22.824^\circ - 22.675^\circ, 74.495^\circ - 74.538^\circ) \Rightarrow (0.149, - 0.043)$$

Non-Euclidean mathematics would ideally be used to find the length of this vector. However, the small angle approximation can be used.

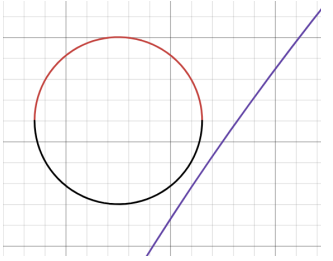
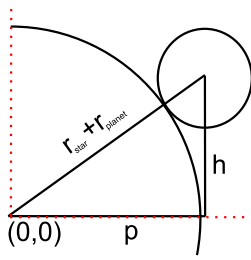
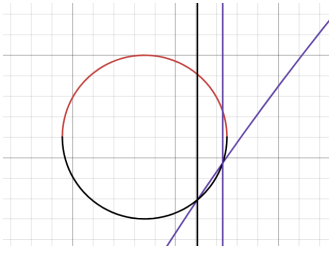
$$\sqrt{0.149^2 + (- 0.0433)^2} = 0.155 = h$$

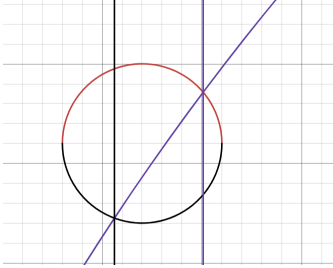
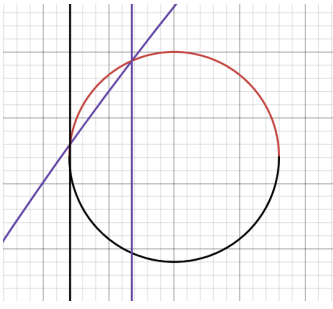
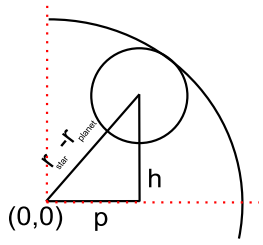
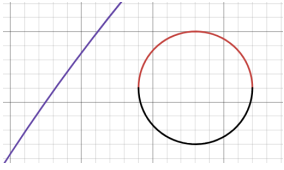
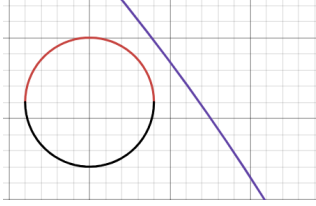
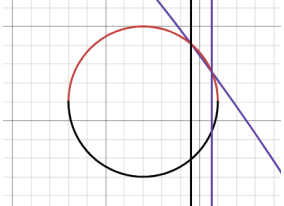
Hence, this is the representation of the transit:



Diagrams 10, 9, and 11. (From left to right: Diagram 10, the mathematical description achieved in the essay plotted on Desmos, Diagram 9, and a real image (Gedzelman, 2020))

The similarity proves the model's accuracy. The following table shows the different phases of the transit and their domain, as the function of the visible area will change. The integration logic comes afterward, as it did not all fit here.

Model image example (Desmos, own source.)	Valid for the domain
	The point at which the diagram and equation become true is the point where there is apparent contact between the objects. This will mark the end of the first domain.  $h^2 + p^2 < (r_{star} + r_{planet})^2$ $p < -\sqrt{-h^2 + (r_{star} + r_{planet})^2}$
	To figure out the limits of this function, the inequation $y < \sqrt{r_{star}^2 - x^2}$ describes how, if the point $(p + r_{planet}, h)$ is inside the star, the integration rule will change again, as the equation describing the upper half of Venus will start occulting, too. Then, the previous function's domain gives the lower end of the domain. $h < \sqrt{r_{star}^2 - (p + r_{planet})^2}$ $-\sqrt{-h^2 + (r_{star} + r_{planet})^2} < p < -\sqrt{-h^2 + r_{star}^2} - r_{planet}$

	The same technique is used here for this part of the model. $y > \sqrt{r_{star}^2 - x^2} \text{ for point } (p - r_{planet}, h)$ $h > \sqrt{r_{star}^2 - (p - r_{planet})^2}$ $-\sqrt{-h^2 + r_{star}^2} - r_{planet} < p < -\sqrt{-h^2 + r_{star}^2} + r_{planet}$
	 $h^2 + p^2 > (r_{star} - r_{planet})^2$ $-\sqrt{-h^2 + r_{star}^2} + r_{planet} < p < -\sqrt{-h^2 + (r_{star} - r_{planet})^2}$
	The fully occulting part will start as follows: $-\sqrt{-h^2 + (r_{star} - r_{planet})^2} < p$
	Furthermore, following the observed symmetry: $-\sqrt{-h^2 + (r_{star} - r_{planet})^2} < p < \sqrt{-h^2 + (r_{star} - r_{planet})^2}$
	$\sqrt{-h^2 + (r_{star} - r_{planet})^2} < p < \sqrt{-h^2 + r_{star}^2} - r_{planet}$

	$\sqrt{-h^2 + r_{star}^2} - r_{planet} < p < \sqrt{-h^2 + r_{star}^2} + r_{planet}$
	$\sqrt{-h^2 + r_{star}^2} + r_{planet} < p < \sqrt{-h^2 + (r_{star} + r_{planet})^2}$
	$\sqrt{-h^2 + (r_{star} + r_{planet})^2} < p$

Now we have the limits, the integration will go as follows, all representing the areas that Venus is blocking or not, in the same fashion as in Diagram 3:

Domains of the function of p	$f(p)$
$-\infty < p$ $p < -\sqrt{-h^2 + (r_{star} + r_{planet})^2}$	$f_1(p) = \pi r_{star}^2$
$-\sqrt{-h^2 + (r_{star} + r_{planet})^2} < p$ $p < -\sqrt{-h^2 + r_{star}^2} - r_{planet}$	$f_2(p) =$ $\pi r_{star}^2 - \int_{x_1}^{x_2} (+\sqrt{r_{star}^2 - x^2})dx + \int_{x_1}^{x_2} (-\sqrt{r_{planet}^2 - (x - p)^2} + h)dx$
$-\sqrt{-h^2 + r_{star}^2} - r_{planet} < p$ $p < -\sqrt{-h^2 + r_{star}^2} + r_{planet}$	$f_3(p) =$ $\pi r_{star}^2 - \int_{x_1}^{x_2} (+\sqrt{r_{star}^2 - x^2})dx + \int_{x_1}^{x_2} (-\sqrt{r_{planet}^2 - (x - p)^2} + h)dx$

	$- 2 \int_{x_2}^{p+r_{planet}} (\sqrt{r_{planet}^2 - (x-p)^2}) dx$
$-\sqrt{-h^2 + r_{star}^2} + r_{planet} < p$ $p < -\sqrt{-h^2 + (r_{star} - r_{planet})^2}$	$f_4(p) =$ $\pi r_{star}^2 - \pi r_{planet}^2 + \int_{x_1}^{x_2} (\sqrt{r_{planet}^2 - (x-p)^2} + h) dx$ $- \int_{x_1}^{x_2} (\sqrt{r_{star}^2 - x^2}) dx$
$-\sqrt{-h^2 + (r_{star} - r_{planet})^2} < p$ $p < \sqrt{-h^2 + (r_{star} - r_{planet})^2}$	$f_5(p) =$ $\pi r_{star}^2 - \pi r_{planet}^2$
$\sqrt{-h^2 + (r_{star} - r_{planet})^2} < p$ $p < \sqrt{-h^2 + r_{star}^2} - r_{planet}$	$f_6(p) =$ $\pi r_{star}^2 - \pi r_{planet}^2 + \int_{x_1}^{x_2} (\sqrt{r_{planet}^2 - (x-p)^2} + h) dx$ $- \int_{x_1}^{x_2} (\sqrt{r_{star}^2 - x^2}) dx$
$\sqrt{-h^2 + r_{star}^2} - r_{planet} < p$ $p < \sqrt{-h^2 + r_{star}^2} + r_{planet}$	$f_7(p) =$ $\pi r_{star}^2 - \int_{x_1}^{x_2} (\sqrt{r_{star}^2 - x^2}) dx + \int_{x_1}^{x_2} (-\sqrt{r_{planet}^2 - (x-p)^2} + h) dx$ $- 2 \int_{p-r_{planet}}^{x_1} (\sqrt{r_{planet}^2 - (x-p)^2} + h) dx$
$\sqrt{-h^2 + r_{star}^2} + r_{planet} < p$	$f_8(p) =$

$p < \sqrt{-h^2 + (r_{star} + r_{planet})^2}$	$\pi r_{star}^2 - \int_{x_1}^{x_2} (+\sqrt{r_{star}^2 - x^2})dx + \int_{x_1}^{x_2} (-\sqrt{r_{planet}^2 - (x - p)^2} + h)dx$
$\sqrt{-h^2 + (r_{star} + r_{planet})^2} < p$ $p < \infty$	$f_9(p) = \pi r_{star}^2$

Giving an area/position graph that looks like this:

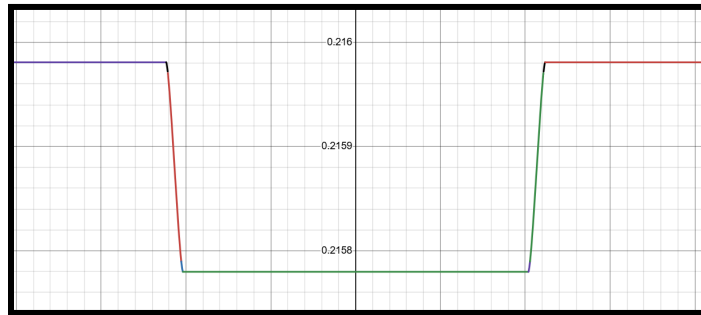


Diagram 12. Desmos, own source.

For this to be comparable to a light curve, the proportional change in area will be used for the y-axis and compared to the proportional change in brightness. The position will be changed to time by dividing it by the velocity at which Venus moves. The movement will be simplified into a straight, uniform, linear movement. This assumption is reasonable as the 7.5 hours the transit lasts represents 0.5° of orbit, allowing for a small angle approximation. Using this assumption, Diagram 9, and the model created, one can estimate the angular velocity:

$$\frac{\Delta p}{\Delta t} = v$$

$$\frac{\text{final position} - \text{initial position}}{\text{final time} - \text{initial time}} = \frac{(0.223378511948 - (-0.223378511948))}{04:49:35(+1 \text{ day}) - 22:09:38} = \frac{0.44676}{23997} = v_{(\text{degrees/second})}$$

$$p * \frac{23997}{0.44676} + 01:29:36 = t$$

The results coincide with Diagram 9.

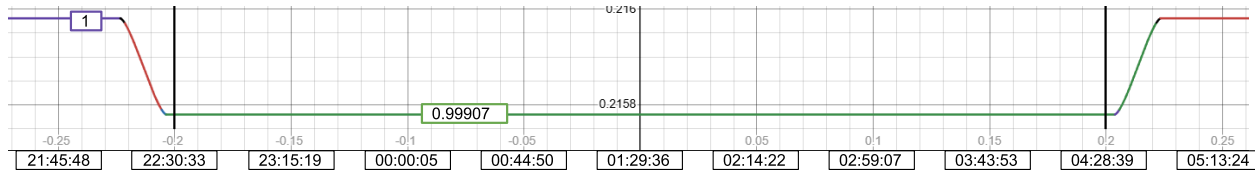
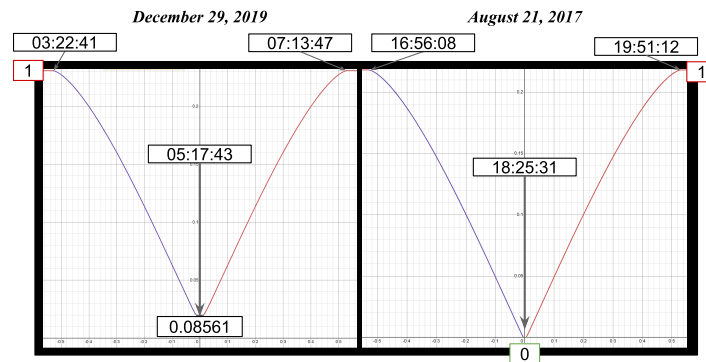


Diagram 13. Desmos, own source.

Interpretation of the results

This essay has produced three results: the annular eclipse of December 26, 2019, and the total eclipse of August 21, 2017.



Diagrams 6 and 7. Area-position graph of the annular eclipse, Area-position graph of the total eclipse.

These two are presented as a display of how the simple model works. They have been deemed as not worth considering for the interpretation of results due to a lack of data on these events. The data found is on the ambient light, which has more varied sources during those events and, in both cases, from locations significantly far from the locations modeled. (Gokhale & Kelkar, 2021) (Jeffrey, R. C., 2017). The transit of Venus is an exciting model that has been investigated in formal research and is comparable to more accurate data. (*NASA - 2012 Transit of Venus*, n.d.) This event was seen with similar characteristics in every part of the world. This has made the results more generalizable, allowing the model to consider the time in UTC. Comparing it to Diagram 9, the timings coincide with good precision.

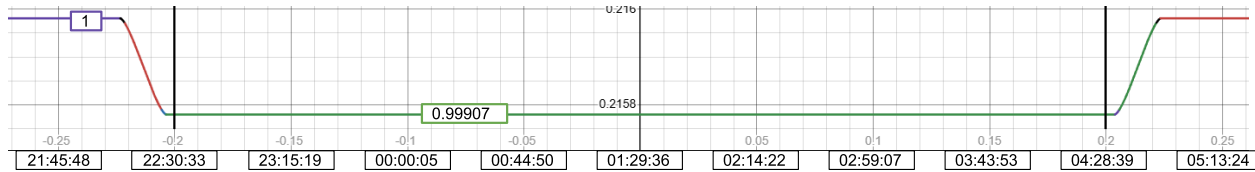


Diagram 13 (Desmos, own source.)

Now, comparing it with the light (Bryant, 2013)

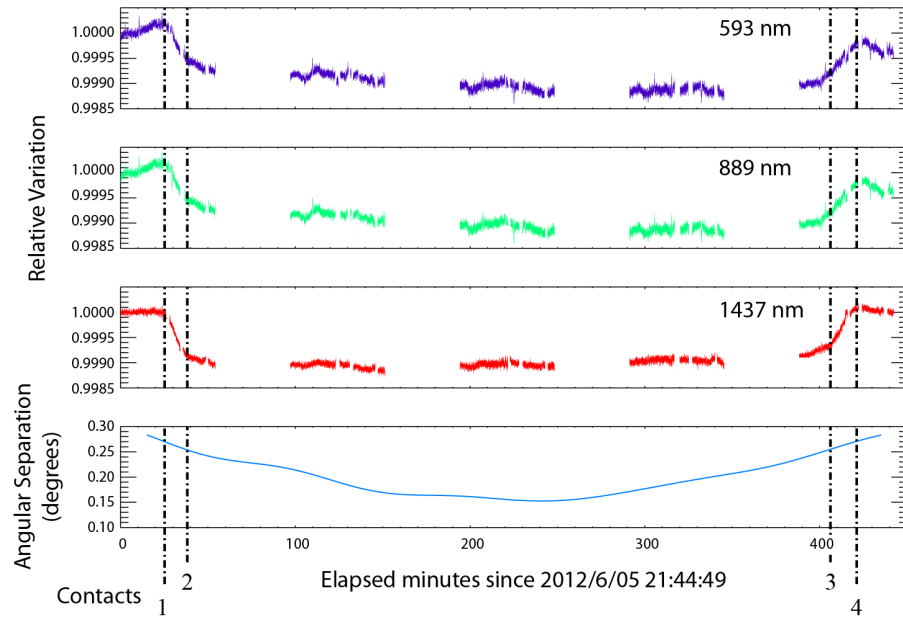


Diagram 14, (Bryant, 2013)

One of the significant successes of this model is the correlation between the light measured in all wavelengths and the proportion of the visible area. As shown in the graphs, the area decreases to 0.99907. In the actual data, it drops by a similar amount. The timings remain accurate, and the shape of the light curve is reasonably similar for all wavelengths. This model demonstrates a good level of accuracy achieved through simple math. Not only this, but linking back to eclipses, the smaller angular radius of Venus in comparison to the Moon should lead to an increase in the errors and inaccuracies that will be discussed in limitations, suggesting the correlation found could argued to be more assertive with eclipses if we were to find adequate data.

Limitations

One limitation of the correlation between brightness and area is that it does not consider the sun's corona, which would prevent the brightness from ever reaching 0, unlike the model's area for the total eclipse predictions. On the Venus transit, the results have two visible effects that cause flaws. Firstly, the effects of limb darkening. Limb darkening is a characteristic of stars that makes their brightness not a perfect circle but exhibits a darkening towards the outer visual radius (Neilson et al., 2017). Diagram 11 provides a good visual insight into the effect. Because of this, when an occulting object blocks the inner part of the star, it has a more significant effect on the brightness than when it blocks a part of the outer rim. This causes the light curve to be bent, in contrast to the utterly flat area calculated, resulting in a worse brightness-area correlation.

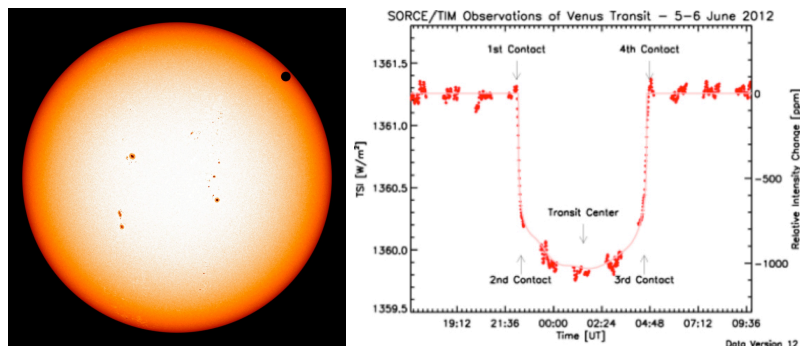


Diagram 11 (Gedzelman, 2020), Diagram 15 (Bryant, 2013)

Another limitation of the model is that in real-world data, one can spot a slight increase in brightness before and after the dip. This is caused by gravitational lensing. (Kasuya et al., 2011)

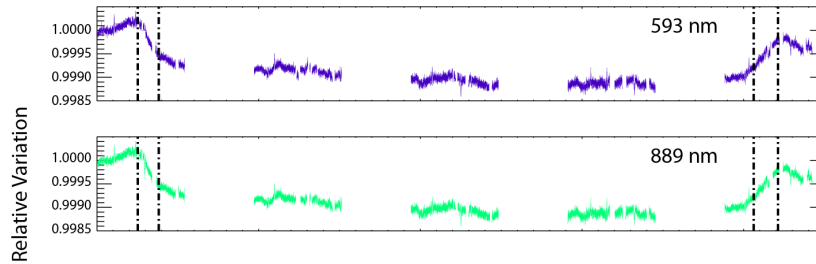


Diagram 14 (Bryant, 2013)

This image illustrates the slight increase, which is more visible in these two wavelengths, right before the planet transits the star. This effect cannot be predicted solely by calculating the star's area, and proper modeling would imply a lot of very high-level physics and math.

Conclusion

This essay's objective is to model the visible area of the sun during occulting solar events and relate it to the sun's brightness. This has been modeled for both total and annular eclipses and a Venus transit; the model can also accommodate partial ones, but they did not fit within the size of the essay. Despite this, the analysis has been thorough, and the case of the Venus transit has produced great results, proving a strong correlation between the visible area and the sun's brightness. Most of the differences and areas where this correlation was weaker were due to effects that have been identified and discussed, which are properties of light or stars that cannot be described through the visible area of a star. While this limitation reflects inaccuracy with the real-life data, the potential for future refinement of the model by taking those effects into account makes it a strong base for studying them. Instead of simply modeling the sun's visible area, a more comprehensive model could include a way to simulate the intensity of brightness in each area, allowing for an analysis of the sun's limb darkening and the corona while also studying the physical phenomena of gravitational lensing. Developing a more precise model could be helpful

in the repeatedly mentioned sister field of exoplanet study, where light curves play a vital role. However, linking back to the original question, the objective of this essay was to find the relationship between data and the area, which was achieved successfully. Overall, the essay has provided fundamental insights and a highly generalizable model for all occultation events. On a more personal note, the production of a mathematical model is something that I was terrified of, as it involved doing something I had never done before; I was more accustomed to performing regression fits. This IA has helped me develop a thorough understanding of mathematical skills and made me aware of the power a mathematical model can have in describing physical phenomena.

References

- Adkins, Frankie. *Why Some Animals Get Agitated During a Solar Eclipse*. 8 Apr. 2024, www.bbc.com/future/article/20240405-solar-eclipse-2024-effect-on-animals.
- Bryant, M. (2013, July 25). *Observations of the Venus Transit - Total irradiance*. SORCE. <https://lasp.colorado.edu/sorce/2012/06/10/observations-of-the-venus-transit-total-irradiance/>
- Caltech. (n.d.). *Data set help: Amateur light curve background information*. <https://exoplanetarchive.ipac.caltech.edu/docs/datasethelp/AXA.html>
- Espenak, F. (n.d.-a). *EclipseWise - Annular Solar Eclipse of 2019 Dec 26*. Fred Espenak. <https://eclipsewise.com/solar/SEprime/2001-2100/SE2019Dec26Aprime.html>
- Espenak, F. (n.d.-b). *EclipseWise - Total Solar Eclipse of 2017 Aug 21*. Fred Espenak. <https://www.eclipsewise.com/solar/SEprime/2001-2100/SE2017Aug21Tprime.html>

- Gedzelman, S. D. (2020, July). *Solar eclipse skies and limb reddening*. Researchgate. Retrieved December 21, 2024, from https://www.researchgate.net/publication/342228780_Solar_eclipse_skies_and_limb_reddening
- Gokhale, S., & Kelkar, A. S. (2021). Effect of solar eclipse on solar radiations in different wavelengths and Earth's atmospheric parameters. *ResearchGate*. https://www.researchgate.net/publication/349883311_Effect_of_Solar_Eclipse_on_Solar_Radiations_in_Different_Wavelengths_and_Earth's_Atmospheric_Parameters_Observed_during_Annular_Solar_Eclipse_in_South_India_on_26_December_2019
- History of Eclipses - NASA Science*. (n.d.). <https://science.nasa.gov/eclipses/history/>
- Jeffrey, R. C. *Eclipse Images by Jeffrey R. Charles*. (2017). <http://www.eclipsechaser.com/eclink/image/total17.htm>
- Kasuya, S., Honda, M., & Mishima, R. (2011, February 18). *New observable for gravitational lensing effects during transits*. Oxford Academic. Retrieved December 21, 2024, from <https://academic.oup.com/mnras/article/411/3/1863/972908>
- Libretexts. (2022, April 11). *18.3: Diameters of Stars*. Physics LibreTexts. [https://phys.libretexts.org/Bookshelves/Astronomy__Cosmology/Astronomy_1e_\(OpenStax\)/18%3A_The_Stars_-_A_Celestial_Census/18.03%3A_Diameters_of_Stars](https://phys.libretexts.org/Bookshelves/Astronomy__Cosmology/Astronomy_1e_(OpenStax)/18%3A_The_Stars_-_A_Celestial_Census/18.03%3A_Diameters_of_Stars)
- NASA - 2012 Transit of Venus*. (n.d.). <https://eclipse.gsfc.nasa.gov/OH/transit12.html>
- Neilson, H. R., McNeil, J. T., Ignace, R., & Lester, J. B. (2017). Limb Darkening and Planetary Transits: Testing Center-to-Limb Intensity Variations and Limb-darkening Directly from Model Stellar Atmospheres. *The Astrophysical Journal*, 845(1), 65. <https://doi.org/10.3847/1538-4357/aa7edf>

Stellarium contributors (2024). Stellarium v24.4 Astronomy Software.

URL <https://stellarium.org/>.

The Editors of Encyclopaedia Britannica. (n.d.). *Understanding the lunar and solar eclipse*

phenomena | *Britannica.* Encyclopedia Britannica.

<https://www.britannica.com/summary/eclipse>

Appendix

Appendix 1

$$A = \pi r^2 - \int_{-r_{star}}^k (+\sqrt{r_{star}^2 - x^2})dx + \int_{-r_{star}}^k (-\sqrt{r_{star}^2 - x^2})dx -$$

$$\int_k^{t+p} (+\sqrt{r_{planet}^2 - (x - p)^2})dx + \int_k^{t+p} (-\sqrt{r_{planet}^2 - (x - p)^2})dx$$

$$\text{for } \{-(r_{star} + r_{planet}) < p < r_{planet} - r_{star}\}$$

Appendix 2

$$\pm \sqrt{r_{planet}^2 - (x - p)^2} + h = \pm \sqrt{r_{star}^2 - x^2}$$

$$\sqrt{r_{planet}^2 - (x - p)^2} = \sqrt{r_{star}^2 - x^2} - h$$

$$r_{planet}^2 - (x - p)^2 = (\sqrt{r_{star}^2 - x^2} - h)^2$$

$$r_{planet}^2 - (x - p)^2 = (r_{star}^2 - x^2) - 2h\sqrt{r_{star}^2 - x^2} + h^2$$

$$r_{planet}^2 - x^2 + 2xp - p^2 = r_{star}^2 - x^2 - 2h\sqrt{r_{star}^2 - x^2} + h^2$$

$$r_{planet}^2 - x^2 + 2xp - p^2 - r_{star}^2 + x^2 - h^2 = -2h\sqrt{r_{star}^2 - x^2}$$

$$\frac{r_{planet}^2 + 2xp - p^2 - r_{star}^2 - h^2}{-2h} = \sqrt{r_{star}^2 - x^2}$$

$$\left(\frac{r_{planet}^2 + 2xp - p^2 - r_{star}^2 - h^2}{-2h}\right)^2 = r_{star}^2 - x^2$$

$$\frac{(r_{planet}^2 + 2xp - p^2 - r_{star}^2 - h^2)^2}{4h^2} = r_{star}^2 - x^2$$

$$(r_{planet}^2 + 2xp - p^2 - r_{star}^2 - h^2)^2 = (r_{star}^2 - x^2) * 4h^2$$

$$(2xp + r_{planet}^2 - p^2 - r_{star}^2 - h^2)^2 = 4h^2 r_{star}^2 - 4h^2 x^2$$

Now, the expansion of the square will follow the form $(j + i)^2 = j^2 + 2ji + i^2$ where $j = 2xp$

and $i = (r_{planet}^2 - p^2 - r_{star}^2 - h^2)$, chosen to isolate x :

$$\begin{aligned} (r_{planet}^2 + 2xp - p^2 - r_{star}^2 - h^2)^2 &= \\ &= 4x^2 p^2 + x * (4p(r_{planet}^2 - p^2 - r_{star}^2 - h^2)) + (r_{planet}^2 - p^2 - r_{star}^2 - h^2)^2 = \\ &= 4h^2 r_{star}^2 - 4h^2 x^2 \end{aligned}$$

$$\begin{aligned} x^2 * 4(p^2 + h^2) + x * 4p(r_{planet}^2 - p^2 - r_{star}^2 - h^2) + \\ + (r_{planet}^2 - p^2 - r_{star}^2 - h^2)^2 - 4h^2 r_{star}^2 &= 0 \end{aligned}$$

This means that, by using the quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Where:

$$a = 4(p^2 + h^2)$$

$$b = 4p(r_{planet}^2 - p^2 - r_{star}^2 - h^2)$$

$$c = (r_{planet}^2 - p^2 - r_{star}^2 - h^2)^2 - 4h^2 r_{star}^2$$

The two solutions are:

$$x_1 = \frac{-4p(r_{planet}^2 - p^2 - r_{star}^2 - h^2) - \sqrt{(4p(r_{planet}^2 - p^2 - r_{star}^2 - h^2))^2 - 16*(p^2 + h^2)((r_{planet}^2 - p^2 - r_{star}^2 - h^2)^2 - 4h^2 r_{star}^2)}}{8(p^2 + h^2)}$$

$$x_2 = \frac{-4p(r_{planet}^2 - p^2 - r_{star}^2 - h^2) + \sqrt{(4p(r_{planet}^2 - p^2 - r_{star}^2 - h^2))^2 - 16*(p^2 + h^2)((r_{planet}^2 - p^2 - r_{star}^2 - h^2)^2 - 4h^2 r_{star}^2)}}{8(p^2 + h^2)}$$

Slightly simplified:

$$x_1 = \frac{-p(r_{planet}^2 - p^2 - r_{star}^2 - h^2) - \sqrt{(p(r_{planet}^2 - p^2 - r_{star}^2 - h^2))^2 - (p^2 + h^2)((r_{planet}^2 - p^2 - r_{star}^2 - h^2)^2 - 4h^2 r_{star}^2)}}{2(p^2 + h^2)}$$

$$x_2 = \frac{-p(r_{planet}^2 - p^2 - r_{star}^2 - h^2) + \sqrt{(p(r_{planet}^2 - p^2 - r_{star}^2 - h^2))^2 - (p^2 + h^2)((r_{planet}^2 - p^2 - r_{star}^2 - h^2)^2 - 4h^2 r_{star}^2)}}{2(p^2 + h^2)}$$

Appendix 3

This last appendix was not submitted to the IB and is a later addition only posted on my website to show all of my work. Please take this last example with caution, as it's not been revised as thoroughly as the rest of the IA since it was never submitted and lacks comparison to real data. It's only a last example of a plausible use of the mathematical model.

=====

Lastly, a more simple partial solar eclipse. One of the problems with these eclipses is that they are not as valuable scientifically as other eclipses and hence the data on them is much smaller. However, given how the model has worked previously, the model would ideally roughly predict the brightness received on the earth during the partial eclipse with only the information that can be extracted from sources such as Stellarium.

The specific example is the eclipse of the 25th of October of 2022 as seen from Lat. N53° 33' 2.63'' Lon E9° 59' 34.87'' (Hamburg, Germany, EU) at an altitude of 0m. The data used has been sourced from (<https://stellarium.org/>), and the point chosen is an example of an eclipse that is partial and not close to being possibly simplified as an annular eclipse.

October 25, 2022 eclipse as seen from *Hamburg, Germany, EU*.

Moon's apparent radius: $0^{\circ}15'57.25'' = 0.2659^{\circ}$

Sun's apparent radius: $0^{\circ}16'04.55'' = 0.2679^{\circ}$

First contact: 09:07:43 UTC

End of the transit: 11:12:48 UTC

Hence, the point of greatest transit (As the essay assumes the movement in front of the star to be a constant one) is the half-time between the start and the end, which is:

10h10m15.5s

Putting this time into Stellarium, and their transformation into decimal degrees is the following

(1 hour = 15 degrees, 60 minutes = 1 hour, and 60 seconds = 1 minute)

Moon's position RA/Dec: 13h59m58.61s/-11°54'36.6''

$$RA: 13 * 15 + \frac{59}{60} * 15 + \frac{58.61}{60^2} * 15 = 209.99421^{\circ}$$

$$Dec: -11 - \frac{54}{60} - \frac{36.6}{60^2} = -11.91^{\circ}$$

Sun's position RA/Dec: 13h59m12.62s/-12°09'42.1''

$$RA: 13 * 15 + \frac{59}{60} * 15 + \frac{12.62}{60^2} * 15 = 209.80258^{\circ}$$

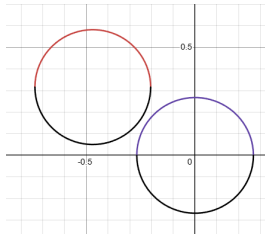
$$Dec: -12 - \frac{09}{60} - \frac{42.1}{60^2} = -12.161694$$

Because we are looking at the position of greatest transit, the distance between the centers of the two bodies will be the smallest possible, hence equal to h , which means that by using the Pythagorean theorem as in the previous example:

$$\begin{aligned} \sqrt{(209.99421 - 209.80258)^2 + (11.91 - 12.161694)^2} &= \sqrt{(0.19163)^2 + (0.25169)^2} = \\ &= 0.31634 = h \end{aligned}$$

Having this information, fully gathered from Stellarium, the same method can be used to extract the domains and the definite integrals.

Image example	Limit and area function
---------------	-------------------------



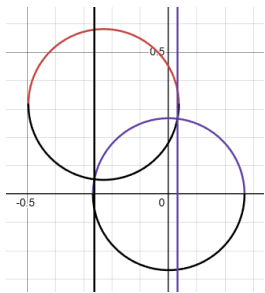
Again reaching out to past logic, the Pythagorean theorem sets the start of the position where the contact begins.

$$-\infty < p$$

$$p < -\sqrt{-h^2 + (r_{star} + r_{planet})^2}$$

And with the area being described by the function:

$$f_1(p) = \pi r_{star}^2$$



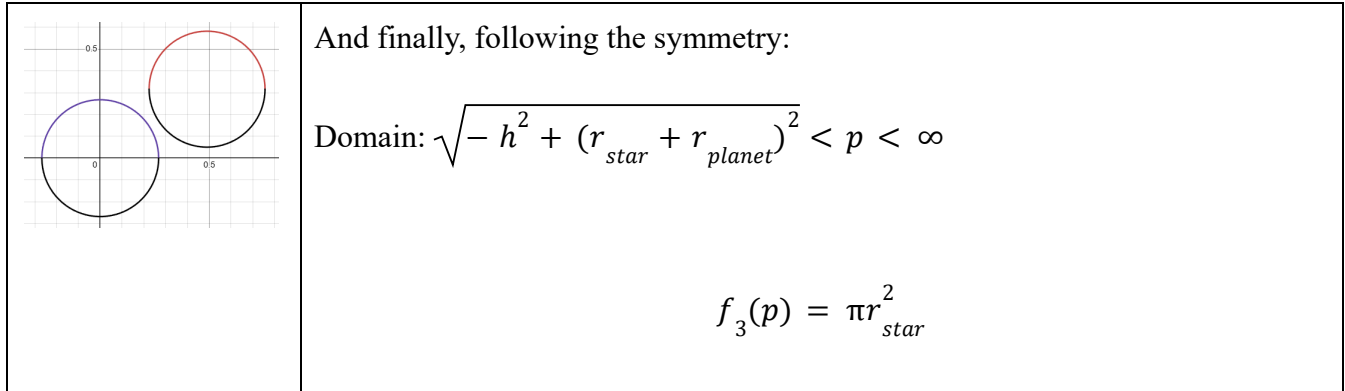
For this case, given how the interaction is all the time between the top part of the star and the bottom part of the moon, there is only one equation to describe the whole transit, as the same two functions interact all the time. This means that the limits are the two starts of the transit.

$$-\sqrt{-h^2 + (r_{star} + r_{planet})^2} < p$$

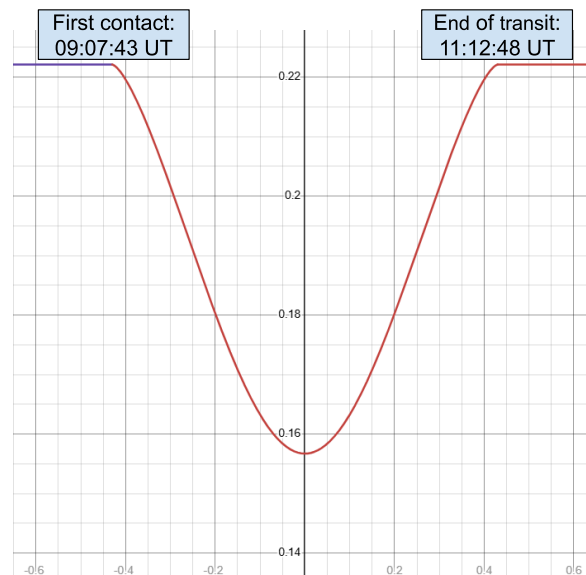
$$p < \sqrt{-h^2 + (r_{star} + r_{planet})^2}$$

And the area:

$$f_2(p) = \pi r_{star}^2 - \int_{x_1}^{x_2} (\sqrt{r_{star}^2 - x^2}) dx + \int_{x_1}^{x_2} (-\sqrt{r_{planet}^2 - (x - p)^2} + h) dx$$



Resulting in a position/area curve that looks like this:



While, as mentioned, there is no real-life data this light curve can be compared to, it comes to show an example of how this could predict future events or help understand what happened in past events that were not measured. To fully develop the result, the relative variation in area could be applied.